— also x an Stelle von $\bar{x}$ —, oder $v > c$ zuläßt; beides führt zu den großen Abweichungen im asymptotischen Gebiet. Wie jedoch Abb. 1 zeigt, erhält man eine recht gute Approximation der relativistischen Diffusionskoeffizienten, wenn man die klassischen Werte durch einen Faktor für $\bar{x}=0$ anpaßt [Gl. (46)] und für

$$\bar{x} > \bar{x}_0 = [m\, c^2/2\, k\, T]^{1/2} \tag{49}$$

konstant gleich dem Wert für $\bar{x}=\bar{x}_0$ setzt, im Zwischengebiet jedoch $\bar{x}$ aus x durch $v=\bar{v}$ berechnet.

Die Reibungskoeffizienten in Abb. 4 stimmen für $x \to 0$ natürlich überein — ein ruhendes Teilchen erfährt in einem ruhenden Medium keine Reibung. Für größere als thermische Energien läßt sich die relativistische Kurve jedoch nur ganz grob nach obigem Verfahren approximieren. Im asymptotischen Bereich gilt

$$\frac{\langle(\Delta u)\rangle}{\langle(\Delta u)\rangle_0} \approx \lim_{x\to\infty} \frac{\langle(\Delta u)\rangle}{\langle(\Delta u)\rangle_0} = \sqrt{\frac{\gamma}{2\,\pi}} \cdot \frac{e^\gamma}{g(\gamma)} \cdot \frac{K_1(\gamma)}{\gamma}\,. \tag{50}$$

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On the Upper Bound for the Ratio of the First Two Membrane Eigenvalues

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Dedicated to Prof. L. BIERMANN *on his 60th birthday*

Let λ_1 and λ_2 ($\lambda_1 < \lambda_2$) be the two lowest eigenvalues of a membrane. We show that the inequality $\lambda_2/\lambda_1 < 2.6578$ is always fulfilled, irrespective of the shape of the membrane. This is an improvement upon results due to BRANDS, PAYNE, PÓLYA and WEINBERGER.

1. Let D be a bounded domain in the x,y-plane with sufficiently smooth boundary C. We consider the membrane eigenvalue problem

$$u_{xx} + u_{yy} + \lambda\, u = 0 \quad \text{in } D\,, \tag{1}$$

$$u = 0 \quad \text{on } C\,. \tag{2}$$

There are infinitely many positive eigenvalues λ_i:

$$0 < \lambda_1 < \lambda_2 \leqq \dots\,.$$

Let us define

$$\nu := \lambda_2/\lambda_1\,. \tag{3}$$

PAYNE, PÓLYA and WEINBERGER [1] habe shown among many other things that $\nu \leqq 3$. Recently BRANDS [2] sharpened this result to $\nu \leqq 2.6861\ldots$ by introducing a parameter α into their method. In this note we shall show $\nu < 2.6578$ by introducing another parameter β into the method of BRANDS. This gain seems to be small, but it should be born in mind that $\nu(\text{circle}) = 2.539$, while $\nu(\text{square}) = 2.500$.

2. For abbreviation we shall use the notation

$$\int f := \iint_D f(x,y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

The eigenfunction corresponding to λ_1 is denoted by u. It is posible to normalize u such that $u > 0$ in D and $\int u^2 = 1$. Now it is known that

$$\lambda_2 \leqq \int (\varphi_x{}^2 + \varphi_y{}^2) \,/ \int \varphi^2, \tag{4}$$

where φ is a sufficiently smooth trial function in $D+C$, being zero on C and orthogonal to u:

$$\varphi = 0 \quad \text{on } C\,, \tag{5}$$

$$\int u\, \varphi = 0\,. \tag{6}$$

As in [1] and [2], (5) is attained by taking multiples of u for φ. We choose as trial functions

$$\varphi := x\, u^\beta, \tag{7}$$

and

$$\varphi := y\, u^\beta. \tag{8}$$

Here β ($\beta \geqq 1$) is a parameter which is later on taken in an optimal way. As in [1] and [2], (6) is attained by taking as origin of the rectangular coordinate system the center of gravity of D with mass distribution $u^{\beta+1}$:

$$\int x\, u^{\beta+1} = \int y\, u^{\beta+1} = 0\,.$$

[1] L. E. PAYNE, G. PÓLYA, and H. F. WEINBERGER, J. Math. Phys. **35**, 289 [1956].

[2] J. J. A. M. BRANDS, Arch. Rational Mech. Anal. **16**, 265 [1964].

3. By partial integration and by the use of (1) and (2), it is easy to verify the following formulas:

$$\beta \int 2\,x\,u^{2\beta-1}\,u_x = -\int u^{2\beta}, \tag{9}$$

$$\int u^{\beta}(u_x^2+u_y^2) = \lambda_1 \int u^{\beta+2}/(\beta+1), \tag{10}$$

$$(2\beta-1)\int x^2 u^{2\beta-2}(u_x^2+u_y^2) = \lambda_1 \int x^2 u^{2\beta} + \int u^{2\beta}/\beta. \tag{11}$$

In deriving (11), use has been made of (9). There exists a counterpart of (11) with x and y permuted.

4. If we take (7) as our trial function, then as a consequence of (9) and (11), (4) leads to

$$\lambda_2 \leqq \frac{\beta^2\lambda_1}{2\beta-1} + \frac{\beta}{2\beta-1}\int u^{2\beta}/\int x^2 u^{2\beta},$$

while (8) leads to the corresponding inequality with x replaced by y. Taking the mean of these two inequalities and dividing by λ_1, we find

$$\nu \leqq \frac{\beta^2}{2\beta-1} + \frac{\beta}{2(2\beta-1)} P(\beta)/\lambda_1, \tag{12}$$

where

$$P(\beta) := \int u^{2\beta}[1/\int x^2 u^{2\beta} + 1/\int y^2 u^{2\beta}]. \tag{13}$$

5. Now P can be estimated by SCHWARZ's inequality. Introducing BRAND's parameter α ($\alpha \geqq \beta$), we find by partial integration

$$\int u^{\alpha+1} = -(\alpha+1)\int x\,u^{\beta}\cdot u^{\alpha-\beta}\,u_x,$$

and from here

$$\left(\int u^{\alpha+1}\right)^2 \leqq (\alpha+1)^2 \int x^2 u^{2\beta} \int u^{2(\alpha-\beta)}\,u_x^2.$$

There is a corresponding inequality with x replaced by y. Using (10), we find for P in (13):

$$P(\beta)/\lambda_1 \leqq \frac{(\alpha+1)^2}{2(\alpha-\beta)+1} Q(\beta, \alpha-\beta+1), \tag{14}$$

where

$$Q(\beta,\gamma) := \int u^{2\beta}\int u^{2\gamma}/\left(\int u^{\beta+\gamma}\right)^2.$$

6. In order to estimate Q, we introduce the functions

$$B(\beta,\gamma) := \int u^{\beta+\gamma}/\int u^{\beta+1}\int u^{\gamma+1}, \tag{15}$$

$$A(\beta) := B(\beta,\beta) = \int u^{2\beta}/\left(\int u^{\beta+1}\right)^2. \tag{16}$$

A has already been defined by BRANDS. Then

$$Q(\beta,\gamma) = A(\beta)\,A(\gamma)/B^2(\beta,\gamma). \tag{17}$$

7. Next we will find inequalities for A and B, by which the region D and the eigenfunction u can be eliminated. We consider the trial function

$$\varphi := u^{\beta}\int u^{\gamma+1} - u^{\gamma}\int u^{\beta+1}, \quad \beta \neq \gamma, \tag{18}$$

which fulfills (5) and (6). If we put (18) into (4), use (10), (15) and (16), and collect terms, we find

$$K(\beta)\,A(\beta) + K(\gamma)\,A(\gamma) \leqq 2\,L(\beta,\gamma)\,B(\beta,\gamma), \tag{19}$$

where
$$L(\beta,\gamma) := \nu - \beta\gamma/(\beta+\gamma-1), \tag{20}$$

and
$$K(\beta) := L(\beta,\beta) = \nu - \beta^2/(2\beta-1). \tag{21}$$

Now we restrict β, γ in such a way that $\beta,\gamma \geqq 1$, but $K(\beta), K(\gamma) > 0$. If $\gamma = 1$, (19) will lead us to

$$1 \leqq A(\beta) \leqq (\nu-1)/K(\beta). \tag{22}$$

Correspondingly we have

$$1 \leqq A(\gamma) \leqq (\nu-1)/K(\gamma). \tag{23}$$

By squaring of (19), we get for the Q of (17):

$$Q(\beta,\gamma)\left[\frac{K^2(\beta)\,A(\beta)}{A(\gamma)} + \frac{K^2(\gamma)\,A(\gamma)}{A(\beta)} + 2\,K(\beta)\,K(\gamma)\right] \leqq 4\,L^2(\beta\,\gamma). \tag{24}$$

Finally we eliminate the quotients of the A's with (22) and (23), that is with

$$K(\beta)/(\nu-1) \leqq A(\gamma)/A(\beta),$$
$$K(\gamma)/(\nu-1) \leqq A(\beta)/A(\gamma).$$

We arrive at

$$Q(\beta,\gamma) \leqq R(\beta,\gamma), \tag{25}$$

where

$$R(\beta,\gamma) := \frac{4\,L^2(\beta,\gamma)\,(\nu-1)}{K(\beta)\,K(\gamma)\,[K(\beta)+K(\gamma)+2(\nu-1)]}. \tag{26}$$

8. The final result is taken from (12), (14) and (25):

$$\nu \leqq \frac{\beta^2}{2\beta-1} + \frac{\beta}{2(2\beta-1)}\,\frac{(\alpha+1)^2}{2(\alpha-\beta)+1} R(\beta, \alpha-\beta+1). \tag{27}$$

It follows from (26), (20) and (21) that the right hand side of (27) is a rational function of α, β, and ν. For every pair (α,β) one can find a number $\bar{\nu}(\alpha,\beta)$, such that (27) is equivalent to

$$\nu \leqq \bar{\nu}(\alpha,\beta).$$

We have done this for some pairs (α, β) simply by iteration on the electronic computer G3. By the method of steepest descent we approximately determined a pair (α, β), where $\bar{\nu}(\alpha,\beta)$ has a minimum. The result: For $\alpha = 1.6643$ and $\beta = 1.1465$ we found: $\nu \leqq 2.65776\ldots$.